Teaching Multi-digit Multiplication using Array-based Materials

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This paper describes research on the classroom practices of seven teachers who taught a lesson on multi-digit multiplication using array-based materials. Students' understanding of multi-digit multiplication just prior to the lesson is contrasted with their performance several weeks after the lesson. Differences among the teachers in the ways they taught the lesson are examined in relation to students' subsequent understanding of multiplication. An important issue was the match (or mismatch) between the demands of the lesson and students' understanding of multiplication prior to the lesson. For example, Teacher A carefully scaffolded her students from single-digit to two-digit by two-digit multiplication problems using the dotty arrays and her students made substantial progress in their understanding of multiplication. Teacher C's students did not seem to have a solid understanding of place value or multiplication with small quantities, and did not learn how to use dotty arrays for multi-digit multiplication. Teacher G's students already understood the partitioning processes needed to solve two-digit by two-digit problems prior to the lesson and probably did not benefit much from the lesson on using dotty arrays.

One of the major goals of today's mathematics instruction is to help students understand the structure of mathematics (Lambdin & Walcott, 2007). The greater focus on mathematical structure can be seen in *The New Zealand Curriculum* (Ministry of Education, 2007a). In contrast to the previous curriculum document (Ministry of Education, 1992), where no mention was made of changes in the nature of the thinking or problem solving over year levels, there is a clear progression in the 2007 document (shown in the achievement objectives under Number Strategies) from simple additive strategies with whole numbers and fractions at level 2 (L2), to additive and simple multiplicative strategies with whole numbers, fractions, decimals, and percentages (L3), to a range of multiplicative strategies when operating on whole numbers, and simple linear proportions, including ordering fractions (L4), to reasoning with linear proportions (L5), and applying direct and inverse relationships with linear proportions (L6). These progressions are closely aligned with the Number Framework, a key aspect of New Zealand's Numeracy Development Project (NDP), a major government initiative aimed at improving the teaching of mathematics and raising the achievement of students (see Bobis et al, 2005; Ministry of Education, 2007b).

The work of Mulligan and colleagues supports the idea that students' appreciation of structure and pattern may be at the heart of differences between high and low achievers in mathematics (Bobis, Mulligan & Lowrie, 2008; Mulligan, Mitchelmore, & Prescott, 2004). Mulligan's research shows that low achievers in mathematics do not appear to notice structure and regularity in mathematics, but intervention drawing their attention to structure and pattern can bring about substantial improvement in their mathematics learning.

The literature on multiplicative thinking and reasoning has been growing steadily over the past decade or so. According to Baek (1998), "understanding multiplication is central to knowing mathematics" (p. 151). The importance of multiplication and division understanding is evident in the NCTM Curriculum Focal Points developed in the US (Beckman & Fuson, 2008; Charles & Duckett, 2008; NCTM, n.d.). NCTM (2000a, 2000b) sees multiplicative reasoning as one of three crucial mathematics themes (along with equivalence and computational fluency) that are interwoven through the Content Standards for the middle grades, forming the foundation for proportional reasoning.

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There are several major differences between additive and multiplicative thinking. For example, multiplication and division have *proportional* structure, whereas addition and subtraction have *part-whole* structure (Sophian, 2007). This means that *multiplicative* partitioning must involve equal-sized parts or groups, whereas *additive* partitioning may result in unequal-sized parts. Additive and multiplicative reasoning are closely linked because understanding multiplicative relationships depends on understanding the concept of a unit, and "that is generally developed first in the context of additive reasoning" (Sophian, 2007, p. 103). It is in considering units of quantification other than *one* that the need for multiplicative relations becomes clear – the unit may be a group (eg, a pair, a trio, or another composite unit) or it may be a fractional quantity (eg, one half, one third, etc). Young children often don't understand the importance of keeping units constant, and when doing equal sharing, tend to divide a continuous quantity into a particular number of pieces, while ignoring the size of the pieces (Sophian, 2007).

A variety of definitions for multiplicative thinking and reasoning have appeared in the literature. According to NDP support materials (Ministry of Education, 2007c, p. 3), multiplicative thinking involves:

> constructing and manipulating factors (the numbers being multiplied) in response to a variety of contexts…[and] deriving [unknown results] from known facts using the properties of multiplication and division [commutative, associative, distributive, inverse].

Multiplicative reasoning is far more complex than additive thinking, and can involve processes such as repeated addition; grouping; number line hopping; number line stretching or compressing; folding and layering; branching; making grids or arrays; area; and proportional reasoning. However, it has been argued that "the most flexible and robust interpretation of multiplication is based on a rectangle" (Davis, 2008, p. 88), thus reinforcing the two-dimensionality of multiplication. An area-based interpretation can be used to show how the algorithm for multi-digit whole-number multiplication works, and can be extended to multiplication of decimal fractions and common fractions (Davis, 2008; Young-Loveridge, 2005a, 2005b). In contrast to multiplicative thinking, additive thinking is a *linear* process involving a single dimension. Number line models typically show addition and subtraction as movement either forwards (addition) or backwards (subtraction) along a line. Hence the use of a repeated-addition strategy to solve a multiplication problem is less advanced than one involving partitioning, manipulating, and recombining quantities using the distributive property (see Ministry of Education, 2006). The inclusion of array diagrams as well as number-line models in the NDP framework book (Ministry of Education, 2007b) provide examples of the richer, more flexible models of multiplication and division.

With the increased emphasis on multiplicative thinking have come expectations about when students should be able to use multiplicative structure. In New Zealand, there is an expectation that by the end of Year 8, students are able to reason multiplicatively (Ministry of Education, n.d.). However, evidence suggests that only about one third of year 8 students have good control over multiplicative structures (Young-Loveridge, 2007, 2008). Hence it is important to understand more about how teachers can help their students to become multiplicative thinkers.

The purpose of the present study was to explore the teaching of multi-digit multiplication using array-based materials in order to understand how different approaches by teachers might impact on students' understanding of multiplication.

Method

Participants

Seven female teachers (A to G) working at the Year 7/8 level (11- to 13-year-olds) from four schools (3 intermediate [year 7-8] and one full primary [year 1-8]) agreed to participate in the study. The decile[1] ranking of the schools ranged from 2 (low) to 9 (high), reflecting the wide range of socio-economic backgrounds of the students. Teachers varied in years of teaching experience from approximately 1.5 years to 20 years. Teachers' experience working with the NDP approach ranged from one to seven years. Each teacher chose a group of students with which to work on enhancing multiplicative thinking. A total of 46 students were present for both lessons and assessments.

Procedure

Each classroom was visited twice. At the first visit, the students were given written assessment tasks to complete, with instructions to *"explain how you worked out your answer. Where possible, draw a diagram to help show your thinking."* The eight tasks included three on whole-number multiplication: two that involved deriving answers from information given and known number facts (*If 4 x 30 = 120, what is 4 x 28? If 5 x 9 = 45, what is 5 x 18?*); and one multi-digit multiplication problem (*What is 11 x 99?*). The teacher then taught a lesson on multi-digit multiplication while the researchers observed. A digital audio-recorder with lapel microphone was worn by the teacher to record as much as possible of the language between herself and the students. After the lesson, the researchers talked to the students and later to the teacher about their experiences of the lesson, to explore their perceptions of the lesson. Two to three weeks later, the students were given written assessment tasks related to the lesson. All teachers taught the same lesson based on NDP support materials on teaching multiplication and division (*Cross Products*: *Multiplication with multi-digit numbers using arrays*, see Ministry of Education, 2007c, p. 67-70).

Results

Students' Prior Knowledge

Most (36 of 46) students found the answer to 4 x 28, and all successfully solved 5 x 18 prior to the lesson. Many (22 students) used a rounding and compensation strategy to solve the first problem, deriving their answer by using a combination of the information given and known number facts (eg, 4 x 30 = 120, 4 x 2 = 8, so 4 x 28 = 120 – 8 = 112). The majority of students (29) used a doubling and halving strategy for the second problem (eg, 5 x 9 = 45, so 5 x 18 = 2 x 45 = 90). Some students (7 on each problem) ignored the information given and instead used standard place-value partitioning to work out their answers (4 x 20 = 80, and 4 x 8 = 32, so 80 + 32 = 112; and 5 x 10 = 50, 5 x 8 = 40, so 50 + 40 = 90). Another group (7 and 9, respectively) used the standard vertical algorithm to work out their answers to 4 x 28 and 5 x 18. Students experienced difficulty with the

[1] Each school in New Zealand is assigned a decile ranking between 1 (low) and 10 (high), based on the latest census information about the education and income levels of the adults living in the households of students who attend that school

problem *11 x 99*, with only 20 students getting the correct answer. Five students used a rounding and compensation strategy, taking 11 from 1100 to get 1089. Five students used standard place-value partitioning, adding 99 to 990. Ten students used the traditional vertical algorithm. Four students did not attempt the problem. One notable misconception was to multiply the tens digits and the ones digits, but not cross-multiply tens with ones (ie, 10 x 90 = 900, 1 x 9 = 9, 900 + 9 = 909). Four of the six children in Teacher E's group used this "buggy" strategy. The two other students gave an answer of 999. For example, E2 wrote on his paper "*allways ad* [sic] *1 more 9*", while E4 wrote "*x11 means 1 extra number for the second number that has to be itself*". Four students from other groups also responded with 999, including C4 who wrote "*11 x 9 = 99, 11 x 99 = 999*" suggesting that she too may have been following the "add another 9" rule used by E2 and E4.

The Lesson on Multi-digit Multiplication

There were many commonalities among the seven teachers in the ways they taught the lesson. For example, all but one teacher used a modeling book (a shared recording book for the group) to record discussions with the children, and began by talking about their planned learning intentions for the lesson. Most discussed the nature of multi-digit numbers and associated issues around place value. Although all of the teachers used "dotty arrays" (see Figure 1), some were photocopied onto paper for students to draw on with pencils or marker pens, leaving a permanent record of the process. Others were laminated and students' recording with a whiteboard pen was erased between problems. Having the paper record to refer back to later was an advantage for the teacher and students, as well as for the researchers.

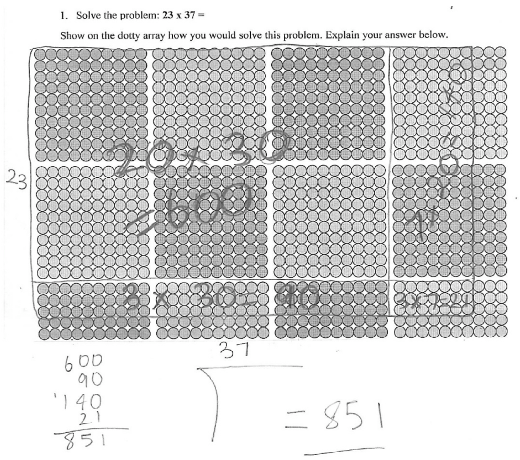

Figure 1. Student B6's record of her solution to 23 x 37 =

Teachers also differed in their approaches in important ways, suggesting that some had accurately matched the demands of the lesson with the learning needs of their students, whereas others had either underestimated or overestimated the demands of the lesson for the group they had chosen to work with (a mismatch). Teachers selecting students whose learning needs were well matched to the lesson tended to begin by introducing arrays using single-digit multiplication (eg, 5 x 6), and this appeared to be helpful for scaffolding the idea of representing multiplication as a rectangle with sides corresponding to each of the factors. Drawing a border around the rectangle formed by the two factors turned out to be important for students' understanding.

Teacher A, the most experienced teacher, started out by asking the students what they noticed about the array, drawing their attention to the regular structure of rows and columns of dots and the separation between each group of ten dots so that the 10 by 10 blocks of 100 dots were easy to see. She then asked them to say how they would show 6 x 5 = 30 on the array, eventually telling them to draw a border around it. In the interview, Teacher A commented in the interview about the importance of drawing a border around the part of the dotty array that represented the problem, saying said "*the border is definitely the key word for me – that's why I underline it* [on the whiteboard]").

The students in Teacher A's group (only one of whom had answered 11 x 99 correctly prior to the lesson), made substantial progress towards understanding multiplication during the lesson. Six of the seven in the group successfully used the dotty arrays to work out the partial products for 23 x 37. Three of them added the partial products to get 851 (and one got 841). Two students did not take the final step of adding the partial products. The only student who did not appear to benefit from the array materials (A3) got partial products of 600, 30, and 70, summing them to 700. She wrote 3 x 10 = 30 (instead of 3 x 30 = 90), only noticed one group of 70, and completely overlooked the 3 x 7. It was interesting to note that for 11 x 99 prior to the lesson, she wrote 110 - 11 = 99, failing to notice that her answer was the same as one of the factors.

Teacher B, who taught the lesson for the first time on the day of the observations, commented that in the future she would start by stressing the importance of drawing a border around the whole problem. However, at least two of her students benefited from their work with the array, progressing from difficulties with 11 x 99 prior to the lesson, to successfully working out 23 x 37 several weeks after the lesson. All but one of Teacher B's students was able to use dotty arrays to work out 23 x 37, although two students failed to add their partial products together at the end. The only student who did not succeed had accidentally drawn his array as 23 x 38, resulting in partial products of 160 and 24 instead of 140 and 21.

Teacher E had the additional challenge of overcoming a "buggy" algorithm (ie, 11 x 99 = 10 x 90 + 1 x 9) taught by another teacher, but two of her students seemed to benefit from the work with dotty arrays. The other students continued to use the vertical written algorithm before drawing borders around unconnected partial products.

Mismatches between Students' Learning Needs and the Lesson

Teacher C chose the lowest of her three groups for the lesson, but may have underestimated how demanding the lesson would be for them. She did not emphasize the importance of drawing a border around the whole problem initially, or lead into the lesson gradually by working with single-digit multiplication problems first. Her students tried to work out the answer to 23 x 37 initially using an algorithmic approach, then fitting the array to that answer, often drawing borders around separate and unconnected cross products. The result was a separate rectangle for each cross product, without connection to the original factors or the total product. Teacher C did not intervene by asking her students to start with a border around the whole problem. However, she was aware that her students had found the lesson difficult, saying "*they found it really, really hard*".

Teacher C's students all appeared to be confused about the use of dotty arrays to solve multi-digit multiplication problems. In the written assessment task completed several weeks after the lesson, not one student was able to work out the answer to 23 x 37. It was evident that most of them had major issues with place value, confusing tens with hundreds, and doing addition instead of multiplication. For example, C1 coloured in three blocks of

100 and a row of seven dots (37), and below this, coloured two blocks of 100 and a row of three dots (23), then "*plused* [sic] *them together*" and wrote her answer of 60 (the sum rather than the product of 23 and 37) (see Figure 2). C2 drew a border around six blocks of 100 (3 across and 2 down), then a separate border around 21 dots in a 3 x 7 array to the right of the block of 600. He added 600 and 21 to get an answer of 621. C3 drew two almost identical arrays to C2, but cut out one block of 100, writing "20 x 30 = 500", then added the 500 and the 21 from the 3 x 7 array to get an answer of 521. Although C4 drew a border around 37 by 23, like C1 she used a highlighting pen to colour in three blocks of 100 and a row of seven dots (37), and below this, coloured two blocks of 100 and a row of three dots (23), giving the answer as 60. C5 drew a border around an array of 30 by 20, and a border around the 7 by 20 array adjacent to the 600. However, there were also borders around 3 by 7, directly below the 30 x 20, and around a 3 by 3 array (instead of 3 x 30) below the 7 by 20 array, resulting in the sum of cross products being 770. C6 drew a border around three blocks of 100, and highlighted a column of three dots on the top left corner of the third block (23). Underneath she drew a border around four blocks of 100, highlighting a column of seven dots on the top left corner of the fourth block (37). She wrote: "*Split the numbers, go back 2 x 3 = 6 and 3 x 7 = 21 then add 10 back onto* [drawing an arrow pointing to the 3 in 2 x 3], *add them together to equal 81*". It was interesting to note that prior to the lesson, C3, C4, and C6 had all given 999 as the answer to 11 x 99 in the initial assessment. Only C2 had correctly partitioned the 99 into 90 and 9 and calculated partial products of 990 and 99 (she made an error in summing these, getting a final answer of 1098).

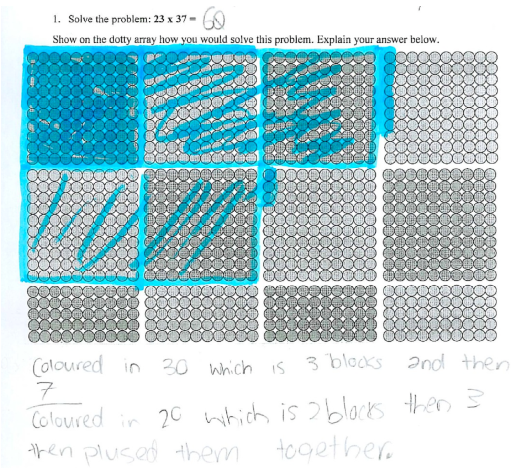

Figure 2. The response of student C1 showing confusion between "tens" and "hundreds," and between addition and multiplication.

It was interesting to note that Teacher C had started straight into the lesson with 23 x 37 without first introducing arrays using single-digit multiplication problems. She accepted her students' strategies of drawing borders around unconnected partial products, and appeared uncertain about how to respond to this. Teacher C's comment that her students "*still struggle a little bit with place value*" reflected her awareness this probably contributed to their difficulty using the arrays. It is possible that Teacher C did not have a solid understanding of the rectangular structure of multiplication herself, and this might explain why she did not recognise the importance of drawing a border around the whole problem initially.

Teacher G chose students who already had a good understanding of multiplication, apparently overestimating their learning needs. Prior to the lesson eight out of nine had solved 11 x 99 successfully. During the lesson, most used the grid method rather than the arrays to work out their answers. They were all able to draw borders and partition the array appropriately and all but one got the correct answer.

Discussion

The findings of this study suggest that arrays can be useful for enhancing students' understanding of multi-digit multiplication, providing there is a good match with their learning needs. Students needed to have good place value understanding in order to partition the two-digit factors into tens and ones and operate with them in the context of multiplication. Dotty arrays enabled the process of multi-digit multiplication to be represented as a rectangle with sides corresponding to the two factors, and this was consistent with Davis' view that, "the most flexible and robust interpretation of multiplication is based on a *rectangle*" (2008, p. 88). The dotty arrays help students to appreciate differences in the magnitude of partial products, and the impact of place value on the size of the sections (ie, partial products) within an array. For example, the six blocks of 100 dots representing 20 x 30 (the "tens") was substantially larger than the 21 dots in the 3 x 7 (the "ones") array. The results reported here support the views of Mulligan and colleagues (see Bobis et al, 2008; Mulligan et al, 2004) that coming to understand the underlying structure of the mathematics is vitally important for effective mathematics learning.

Some students who were already using vertical written algorithms or the "grid" method had difficulty understanding how arrays could be useful for solving multiplication problems. An emphasis on procedural knowledge and rules (without understanding), as reflected in the use of algorithmic approaches to multiplication, may undermine conceptual understanding. As Pesek and Kirschner (2000) have shown, once students have been trained to use standard written algorithms, it can be extremely difficult to then try to help them develop relational understanding. It is rather unfortunate that in the NDP support materials (Book 6), the lesson on traditional written algorithms for multiplication (*Paper Power*) comes *before* rather than *after* the lesson on using dotty arrays to make sense of multi-digit multiplication. The findings of the study are consistent with the idea that teachers' knowledge and understanding of mathematics has a great impact on their teaching (see Ball, Hill & Bass, 2005; Hill, Rowan & Ball, 2005). It is clear that multiplicative reasoning is complex and multi-faceted. There are many challenges for teachers to fully understand the many aspects of multiplicative thinking and then decide on the best ways to support their students in acquiring that conceptual understanding.

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